Nash-Walras Equilibria
Topological Views on Implementation Theory

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Abstract

Nash and Walras equilibria have been studied from the angle of differential topology, yielding comparable results about existence and topological structure of equilibrium sets. Following a presentation of such results we shall review results on the Nash implementation of Walrasian equilibria. The purpose of the exercise is twofold: to establish a refinement criterion among multiple Walrasian equilibria and to study mechanisms closer to real markets than the Walrasian *tâtonnement*. Following up on results relating the best-reply dynamics and the natural projection, our conclusion is that some adjustments on the economic model are likely needed for implementation to link the topological structure of Walrasian and Nash equilibria.

Contents

Introduction

Equilibrium theories are concerned with the study of particular dynamic systems in the Euclidean space. Their crucial objects are the stable points or equilibria of the system. The study of such theories attempts to answer questions such as existence, number of equilibria and stability with respect to perturbations of the underlying parameter space or endogeneity in those parameters (v.g. evolutionary games).

Beyond its epistemic implications, the axiomatic formulation of the theories has given rise to broader and deeper use of mathematics. The study of comparative statics, useful in problems such as existence and number of equilibria, has led to the use of differential and algebraic topology. Since mid-20th century this geometric approach has been applied to two theories relevant to the social sciences: *general equilibrium theory* and *game theory*.

Since John Nash's paper *Equilibrium points in n-person games* appeared in 1950, non-cooperative games have been a standard tool in the study of economics and social sciences (see [My]). This discipline known as classic game theory deals with the neoclassical profit-maximising agent interacting with agents having similar properties. In a normal form game the strategy set is defined as a finite subset of the Euclidean space, where to each N-tuple of individual strategies corresponds an N-dimensional vector payoffs. Each agent wants to maximise his own payoff. The result is a strategy profile where each individually strategy is a best reply against the others, thus leading to a strategically stable outcome.

An important economic concept is that of *efficient* allocation of resources, which is the topic of general equilibrium theory. A market is defined as a set of agents, each of them having a tuple of *initial endowments* and a binary *preference relation* over the commodities. To satisfy their demand for commodities they are allowed to trade. A price system signals at the same time the desire for a commodity and its relative scarceness. At a given price level, initial endowments determine the agent's wealth or budget constraint. The equilibrium of the market is a distribution of the initial endowments that will maximise every agent's preference with respect to the budget constraint. As a consequence, the market is depleted.

Both normal form games and exchange economies can be parametrised in the Euclidean space, and so the question arises about the geometry of the equilibrium sets. It turns out that whereas the set $WE_{\mathscr{E}}$ of market equilibria is a *smooth manifold* diffeomorphic to the space of endowments, the set NE of Nash equilibria is merely homeomorphic to the space of normal form games. The structure induced by the Nash best-response dynamics is not differentiable. Beyond this, a number of similarities can be observed. The first goal of this paper is to present these similar results. To do so we shall adopt the formulation of normal form games from the differentiable viewpoint.

The issue of finding a link between economies and games has a mathematical interest: the topological relation between the Nash manifold and the Walras manifold. There is also a philosophical motivation, the Nash manifold and the Walras manifold being mathematical objects representing central economic concepts. If the same agents that interact strategically in a game are those who determine efficient allocations in the market, it makes sense to think that the Nash equilibrium and the Walrasian equilibrium are both manifestations of economic rationality. To examine the theoretical ground of the consistency between both through implementation theory is our second goal.

The rest of this paper is organised as follows. In section 1 we discuss the Arrow-Debreu model with complete finite markets, providing fundamental results about the topological structure of the equilibrium set. Section 2 introduces classic game theory, providing analogous topological results through the study of differentiable games. Throughout both sections, our aim is to provide a unified framework for section 3.1 to study the implementation of Walrasian equilibria as Nash equilibria. Eventually we shall propose some concluding remarks and some questions raised along the way.

1 Walrasian equilibria

The origins of general equilibrium theory can be traced back to the 19th century, when L. Walras established the basis of neoclassic economic theory. The main concern is efficiency in the allocation of resources or commodities. A *commodity* is characterised by two things: agents want it and it is scarce. According to individual preferences and a set of initial endowments with which they are allowed to trade, agents establish a demand for *consumption bundles*.

Scarceness and demand lead to a set of exchange rates between all pairs of commodities, which in this theory is expressed through a system of *prices*. At given prices, initial endowments can be quantified in terms of a *numeraire* called money, which determines the agent's *wealth*. Subject to this wealth, or budget constraint, each agent seeks to maximise his preference. A pair of prices and endowments such that the corresponding market is depleted (*i.e.* total demand equals total endowments) is called a *Walrasian equilibrium*.

Efficient allocation of resources, understood as a no-waste allocation maximising individual utilities, was Walras' main concern. He named efficient allocations *equilibria*. W. Pareto, who occupied Walras's position in Lausanne after he resigned, went astray from this concept of efficiency. Instead he defined an equilibrium as an allocation such that the only way to improve one agent's welfare is at the cost of another agent's.

Almost a century later K. Arrow proved that under certain assumptions -especially on the form of preference relations- both formulations are equivalent. The Walrasian equilibrium -a concept also referred to in the literature as the *competitive outcome*- thus became the paradigm of efficient allocation.

1.1 Preliminaries

1.1.1 History and definitions

In this section we shall follow the Arrow-Debreu model in pure exchange economies -*i.e.* no production- with *complete* markets, first proposed by K. Arrow and G. Debreu in 1954 (see [AD]). The model was fully studied by Y. Balasko in [Ba88] and summarised in [Ba78]. Let us consider exchange economies with $k < \infty$ commodities and a finite set N of agents.

A vector $x^n \in \mathbb{R}^k$ represents agent n's consumption, where negative quantities are to represent debts. A vector $\omega^n \in \mathbb{R}^k$ denotes agent n's initial endowment. $\Omega = (\mathbb{R}^k)^N$ is the space of endowments. As in the case of games, individual consumption bundles might belong to lower-dimensional spaces. Even though agents might be interested in different commodity bundles, for the sake of simplicity we will systematically embed all consumption spaces in the same Euclidean space.

Axiom 1. Each agent's preference is represented by a binary relation $\preceq^n$ over $\mathbb{R}^k$. Such relation is a linear preordering (*i.e.* reflexive, transitive and total), continuous, monotonic and strictly convex.

Axiom 1 is assumed henceforth. It is proven in [Ba88] that any such preference can be represented by a continuous and strictly concave *utility function* $u^n : \mathbb{R}^k \to \mathbb{R}$.

Axiom 2. For every $n \in N$ the utility function u^n is smooth and differentiably strictly increasing, *i.e.* $\frac{\partial u^n(x)}{\partial x_j}(x) > 0$ for every $j = 1, \cdots, k$.

The use of axiom 2 can be justified in the model using the Stone-Weierstrass approximation theorem (namely that $\mathscr{C}^\infty$ is dense in $\mathscr{C}^0$). Let an economy be denoted by $\mathscr{E} = (u^n)_{n \in N}$. We shall consider in this section economies satisfying axiom 2, restricting our attention to commodities seen as *goods* and not *bads* -although it can also be done-. For that reason prices are in $\mathbb{R}^k_{++}$, unlike section 3.3.2 where prices are allowed to be 0.

Define the individual budget constraint $\mathscr{B}(p, p \cdot \omega^n) = \{x \in \mathbb{R}^k : \ p \cdot x \leq p \cdot \omega^n\}$, which is convex by the linearity of the restriction. It can also be made compact, because not all restrictions on this set are active when the agent maximises $u^n(x)$ subject to $p \cdot x \leq p \cdot \omega^n$.

In [Ba88] it is proven that the problem has a solution, which is unique by the strict concavity of u^n, and so the individual demand function is defined

$$\begin{array}{rccl} f^n: & S \times \mathbb{R} & \to & \mathbb{R}^k \\ & (p, p \cdot \omega^n) & \mapsto & argmax\, \{u^n(x) : x \in \mathscr{B}(p, p \cdot \omega^n)\} \end{array}$$

The monotonicity of utility functions implies that f^n satisfies Walras' law $p \cdot f^n(p, p \cdot \omega^n) = p \cdot \omega^n$, and from the implicit function theorem follows that f^n is smooth. Furthermore the function is homogeneous of degree 0, which implies that the set of prices can be normalised either to the $(k-1)$-simplex or to the $(k-1)$-hyperplane $S = \mathbb{R}^{k-1}_{++} \times \{1\}$, which will be doing. We define the excess demand function

$$\begin{array}{rccl} z: & S \times \Omega & \to & \mathbb{R}^k \\ & (p, \omega) & \mapsto & \sum_{n \in N} (f^n(p, p \cdot \omega^n) - \omega^n) \end{array}$$

Definition 3. $(p, \omega) \in S \times \Omega$ is a Walrasian equilibrium if $z(p, \omega) = 0$. The equilibrium set is denoted by by $WE_{\mathscr{E}} = z^{-1}(0)$.

Walrasian equilibria are parametrised by utility functions, although we will only use the subindex when the dependence needs to be stressed. In a broader context, a set WE can be defined (see the formulation of the model in [M]) as the set of tuples $((u^n)_{n \in N}, p, \omega)$ such that (p, ω) is an equilibrium for the economy $\mathscr{E} = (u^n)_{n \in N}$.

1.1.2 Defining the mechanism: an early proof of existence

Once Walras formulated the concept of equilibrium, he abandoned the problem of existence. However, he still had to provide an idea of how interaction in the market led to prices. The mechanism he came up with was a game of market players against an abstract *auctioneer*. This fictitious player who be in a sense the *invisible hand* of the market.

A detailed explanation of the *tâtonnement* mechanism can be found in [Uz]. Roughly $WE_{\mathscr{E}}$ is the set of stable points of the dynamic system given by $\dot{p} = z(p)$. Let us normalise the set of prices to the $(k-1)$-simplex and consider the following mechanism:

- The auctioneer announces a price vector.
- Each agent announces $f^n(p, p \cdot \omega^n) - \sum \omega^n$
- The auctioneer announces the new price vector $\gamma(p)$ where

$$\begin{array}{rccl} \gamma: & \Delta & \to & \Delta \\ & p & \mapsto & \left(\frac{max\{p_j + z_j(p), 0\}}{\sum_{l=1}^{l=k} max\{p_l + z_l(p), 0\}} \right)_{j=1}^{j=k} \end{array}$$

[Uz] establishes the existence of a solution for $\gamma(p) = p$ using Brower's fixed point theorem (hence tha advantage of normalising prices to a compact space). The stable points of the dynamic system being precisely the Walrasian equilibria ($\gamma(p) = p$ if and only if $\dot{\gamma}(p) = 0$), which yields the existence of equilibrium. We stress that the mechanism depends on a central demand-aggregating agent, and yet there is no such agent in real markets. A mechanism explaining the work of the market is yet to be provided.

1.2 Structure of the equilibrium manifold

The Arrow-Debreu model is formulated in [Ec], where the space of commodities is allowed to be either finite or infinite-dimensional. The paper is about how the techniques involved in the solution of the finite-dimensional model can be extended to infinite dimension. The following sections (except for section 1.5) are based on it. Let us define the restriction π of the canonical projection $\mathbb{R}^k \times \Omega \to \Omega$ to $S \times \Omega$. It follows that $WE_{\mathscr{E}} = \pi^{-1}(\Omega)$.

1.2.1 Notions of differential topology: Euclidean manifolds

A *differentiable manifold* X of dimension n is a subset of the Euclidean space that is locally diffeomorphic to $\mathbb{R}^n$ (and such diffeomorphism is called a *local parametrisation*). The derivative of such diffeomorphism around a point $x \in X$ defines a linear transformation whose image is called the tangent plane at x, $T_x(X)$. For manifolds, derivative functions and tangent planes we refer to [GP].

Definition 4. Let $X \subseteq \mathbb{R}^k$ y $Y \subseteq \mathbb{R}^n$ two differentiable manifolds and $f : X \to Y$ a smooth function. We say f is a *local submersion* in $x \in X$ if $df_x : T_xX \to T_{f(x)}Y$ is surjective (*i.e.* it has full rank).

Theorem 5. Let X and Y be two manifolds of respective dimensions k and l, where $k > l$. Let $f : X \to Y$ be a submersion in x. Then there exist parametrisations $x = \phi(x_1, \dots, x_k)$ and $f(x) = \psi(x_1, \dots, x_l)$ around x and $f(x)$ such that $f(x_1, \dots, x_k) = (x_1, \dots, x_l)$. In other words, f is locally equivalent to the canonical submersion $(x_1, \dots, x_l, \dots, x_k) \mapsto (x_1, \dots, x_l)$.

A proof of this theorem is provided in [GP]. We say x is a *singular point* of f if f is not a local submersion in x, and $y \in Y$ is a regular value of f if it is not the image of a singular point of f. Theorem 5 thus implies the following corollary.

Corollary 6 (The regular value theorem)**.** If $y \in Y$ is a regular value of $f: X \to Y$ then $f^{-1}(y)$ is a differentiable $(dim(X) - dim(Y))$-manifold.

Corollary 7. If $dim(X) = dim(Y)$ then $f^{-1}(y)$ is a finite set for any regular value $y \in Y$.

Proof. $f^{-1}(y)$ is a 0-manifold, i.e. a discrete set with respect to the subspace topology in X. By lemma 10 it is also a compact set. □

1.2.2 The local structure

Theorem 8. $WE_{\mathscr{E}}$ is a $(\sharp N)k$-manifold.

Proof. It is proven in [GP] that a product of two manifolds is also a manifold whose dimension is the sum of the original two, which makes $S \times \Omega$ a $(k - 1 + k(\sharp N))$-manifold. Let

$$\begin{array}{rccl} \bar{z}: & S \times \Omega & \to & \mathbb{R}^{k-1} \\ & (p, \omega) & \mapsto & \sum_{n \in N} (\bar{f}^n(p, p \cdot \omega^n) - \bar{\omega}^n) \end{array}$$

be the truncated excess demand. From Walras' law it follows that $z(p, \omega) = 0$ if and only if $\bar{z}(p, \omega) = 0$, and so $E = \bar{z}^{-1}(0)$. Fixing one agent n, we can see that the Jacobian matrix has a $(k-1) \times k$ submatrix given by the partial derivatives with respect to ω^n for each $n \in N$. By the chain rule this matrix has the form

$$\begin{pmatrix} \frac{\partial f_1^n}{\partial (p \cdot \omega^n)} p_1 - 1 & \frac{\partial f_1^n}{\partial (p \cdot \omega^n)} p_2 & \dots & \frac{\partial f_1^n}{\partial (p \cdot \omega^n)} p_{k-1} & \frac{\partial f_1^n}{\partial (p \cdot \omega^n)} \\ \frac{\partial f_2^n}{\partial (p \cdot \omega^n)} p_1 & \frac{\partial f_2^n}{\partial (p \cdot \omega^n)} p_2 - 1 & \dots & \frac{\partial f_2^n}{\partial (p \cdot \omega^n)} p_{k-1} & \frac{\partial f_2^n}{\partial (p \cdot \omega^n)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_{k-1}^n}{\partial (p \cdot \omega^n)} p_1 & \frac{\partial f_{k-1}^n}{\partial (p \cdot \omega^n)} p_2 & \dots & \frac{\partial f_{k-1}^n}{\partial (p \cdot \omega^n)} p_{k-1} - 1 & \frac{\partial f_{k-1}^n}{\partial (p \cdot \omega^n)} \end{pmatrix}$$

which is equivalent by columns to

$$\begin{pmatrix} -1 & 0 & \dots & & 0 & \frac{\partial f_1^n}{\partial p \cdot \omega^n} \\ 0 & \ddots & & & \vdots & \\ \vdots & & -1 & & & \vdots \\ & & & \ddots & & \\ 0 & 0 & \dots & & -1 & \frac{\partial f_{k-1}^n}{\partial p \cdot \omega^n} \end{pmatrix}$$

whose rank is $k-1$. $d\bar{z}_0$ has full rank, and so $E = \bar{z}^{-1}(0)$ is by theorem 6 a manifold of dimension

$$k - 1 + k(\sharp N) - (k-1) = k(\sharp N). \quad \square$$

1.3 The global structure

Theorem 9. The equilibrium manifold is diffeomorphic to $\Omega = \mathbb{R}^{k(\sharp N)}$.

A short proof of this result is provided in [Ba78]. Note that the space of endowments is precisely $\mathbb{R}^{k(\sharp N)}$, which means that $WE_{\mathscr{E}}$ is diffeomorphic to the space of endowments.

1.4 Regularity and Pareto optima

This section is exposed with greater detail in [Ec].

Lemma 10. $\pi : E \to \Omega$ is *proper*, *i.e.* the inverse image of any compact set is compact.

Theorem 11. The set of regular economies $\mathscr{R}$ is an open dense subset of Ω.

Theorem 11 follows from Sard's theorem, which states that the set of singular equilibria is closed and of Lebesgue measure 0 (see [Ba78]). The restriction of π to $\pi^{-1}(\mathscr{R})$ is a local diffeomorphism, hence a *covering* of $\mathscr{R}$ having a finite number of sheets (see corollary 7). Furthermore, by corollary 7 $\sharp\pi^{-1}(\omega) < \infty$ for $\omega \in \mathscr{R}$. In fact, it is constant over each connected component of $\mathscr{R}$.

Definition 12. The vector $x = (x^1, ..., x^N) \in \Omega$ is a *Pareto optimum* if an agent's utility can only be improved at the cost of another agent's utility, *i.e.* there is no $y \in \Omega$ such that

$$\begin{cases} \sum_{n\in N} x^n = \sum_{n\in N} y^n \\ u^n(y^n) \geq u^n(x^n), \ \forall n \in N \text{ at least one of the inequalities being strict.} \end{cases}$$

Let P be the set of Pareto optima.

Theorem 13 (First Theorem of Welfare Economics)**.** If $(p, \omega) \in WE_{\mathscr{E}}$, it follows $(f^1(p, p\cdot\omega^1), \ldots, f^N(p, p\cdot\omega^N)) \in P$.

Theorem 14 (Second Theorem of Welfare Economics)**.** Let $\omega \in \Omega$ be a Pareto optimum. Then there exists a unique equilibrium price $p \in S$ associated to ω.

For a proof of the theorems see [Ba88] or [Ec]. Let T be the set of no-trade equilibrium allocations and P the set of Pareto optima. We then have the following result.

Corollary 15. $\pi^{-1}(P) \subseteq T$.

From theorem 13 (proven in [Ba88]) and the corollary it follows that $\pi^{-1}(P) = T$, namely that a Pareto optimum is associated to a no-trade equilibrium.

1.5 Topological degree and the number of equilibria

If ω is regular then $d\pi_{(p,\omega)}$ is an isomorphism of finite-dimensional spaces, and is thus representable by an invertible matrix. We define the *Brower degree*

$$deg(\pi,\omega) = \sum_{(p,\omega)\in\pi^{-1}(\omega)} sign\ |d\pi_{(p,\omega)}|$$

which is constant over connected components of $WE_{\mathscr{E}}$ (see [Mi]). We thus define $deg_C(f)$ over connected components C of the domain.

Theorem 16. Let X and Y be two manifolds without boundary of the same dimension. In addition let X be connected, and let $f: X \to Y$ be a proper function. If $deg_X(f) \neq 0$ it follows f is surjective.

Proof. X is connected, and so we can define $deg_X f$. The function being proper, it makes sense to define $\sum_{x\in f^{-1}(y)} sign\ |df_x|$ for a given y. Now, if f is not surjective, there exists $y \in Y$ such that $f^{-1}(y) = \emptyset$, and so trivially $\sum_{x\in f^{-1}(y)} sign\ |df_x| = 0$. □

It is proven in [AH] that the price vector associated to a Pareto optimum is unique, and so the restriction of π to T is a bijection, and in fact a diffeomorphism. From the second-order conditions for the demand functions, [Ba78] proves that Pareto optima are regular equilibrium allocations, hence the result $deg_P(\pi) = 1$.

The uniqueness of the Pareto optimum yields that $|d_\omega(\pi)|$ is either 1 or -1 at any $\omega \in P$, depending on the coordinates chosen for the manifold. A positive orientation can always be chosen in order to guarantee that $deg_C(\pi) = 1$ where C is the connected component that contains P (it is not used here, but [Ba78] proves that P is contained in a connected component of $WE_{\mathscr{E}}$). It is left to extend the result to the whole manifold.

Lemma 17. Let $f, g : X \to Y$ be smooth homotopic maps between manifolds of the same dimension, where X is compact and without boundary. If $y \in Y$ is a regular value for both f and g then

$$\sharp f^{-1}(y) \equiv \sharp g^{-1}(y) \ (mod\ 2).$$

The previous lemma is proven in [Mi] using the fact that compact 1-manifolds always have an even number of boundary points.

Theorem 18. If y and z are regular values of f then

$$\sharp f^{-1}(y) \equiv \sharp g^{-1}(y) \ (mod\ 2)$$

for any g in the smooth *homotopy class* of f.

Proof. Let $h : X \to Y$ be a diffeomorphism which is *isotopic* to the identity and which carries y to z. Now z is a regular value of $h \circ f$, which is homotopic to f. By the previous lemma $\sharp(h \circ f)^{-1}(z) \equiv \sharp f^{-1}(z) \ (mod\ 2)$. Since $(h \circ f)^{-1}(z) = f^{-1}(y)$, the result follows. □

From the theorem we can define the *degree mod 2* of the function π. This number is a constant over all regular equilibria, which are dense by Sard's theorem. It follows that π is surjective over $WE_{\mathscr{E}}$. Furthermore the number of equilibria is generically odd (except for the singular equilibria, which by Sard's theorem have measure 0).

The tools from differential topology that have been discussed and applied through this section rely on the assumption of the manifolds having no boundary. Whereas in the Walrasian setting the set of prices can be normalised to $\mathbb{R}^{k-1}_{++} \times \{1\}$, which has no boundary, in the context of game theory the probability simplex will always lead to sets with boundary in the Euclidean space.

2 The Nash equilibrium manifold in games of complete information

Given a finite set N of players, there is a set S^n of $k_n \in \mathbb{N}$ strategies player $n \in N$ can choose from. Strategy sets might have different sizes. However, we will consider every strategy space to be embedded in $\mathbb{R}^k$ where $k = max\{k_n \mid n \in N\}$. Each pure strategy profile is associated to a vector in $\mathbb{R}^N$ whose coordinates are payoffs to each player. We thus define a game in normal form as parametrised by its payoff structure, and so $\Gamma = \mathbb{R}^S$ for $S = \prod_{n\in N} S^n$.

To allow strategy randomising, or equivalently playing different strategies with a given distribution, we will consider a strategy for player n to be a vector $\sigma^n \in \Sigma^n := \Delta(\mathbb{R}^k)$, the simplex of probability distributions over $\mathbb{R}^k$. A pure strategy s^n for player n can thus be identified to a canonical unit vector.

A strategy profile is an element $\sigma = (\sigma^1, \ldots, \sigma^N) \in \Sigma := (\Delta(\mathbb{R}^k))^N$. *Mixed* strategies yield payoffs within the convex hull of S. However they do not span the convex hull, since players do not correlate their mixed strategies and so they can't reach all possible convex combinations.

Thus Γ is a subset of the finite-dimensional Euclidean space. In a non-cooperative game, a player determines his own actions but cannot influence those of the rest of the players, being thus led to solve a strategic optimisation problem. such solution is the central concept of game theory: the *Nash equilibrium*.

Definition 19. A Nash equilibrium is a profile of strategies which are mutual best replies. We denote

$$NE = \{(G^*, \sigma^*) \in \Gamma \times \Sigma \mid G^{n*}(\sigma^{n*}, \sigma^{-n*}) \geq G^{n*}(\sigma^n, \sigma^{-n*})\ \forall n \in N\ \forall \sigma^n \in \Sigma^n\}.$$

A dynamic formulation of Nash equilibria can be proposed. Given $\Gamma^{-n} = \prod_{j\neq n} \Gamma^j$ and $\sigma^{-n} \in \Gamma^{-n}$ the incomplete strategy profile that excludes player n, let us define

$$\begin{array}{llll} BR: & \Gamma \times \Sigma & \to\to & \Gamma \times \Sigma \\ & (G, (\sigma^1, \ldots, \sigma^N)) & \mapsto\mapsto & (G, \prod_{n\in N} argmax\{G^n(\tau^n, \sigma^{-n}):\ \tau^n \in \Sigma^n\}) \end{array}$$

Notice that the set of maximising arguments for player n is not necessarily a singleton, which is why BR is not a function. Having defined the correspondence, it follows that NE is the set of fixed points of *BR*. In such an approach, the geometry of the equilibrium set provides information about the comparative statics at equilibria.

2.1 The structure of the equilibrium set

Let us define π as the restriction of the natural projection $\Pi : \Gamma \times \Sigma \to \Gamma$ to NE.

Theorem 20. NE is homeomorphic to Γ. Moreover, the following diagram commutes

$$\begin{array}{ccc} \Gamma & \xrightarrow{\phi} & NE \\ & \searrow^{id} & \downarrow \pi \\ & & \Gamma \end{array}$$

The theorem, proven in [KM], states that NE and Γ are topologically undistinguishable. Since a homeomorphism is a bijection, the theorem implies the existence of an equilibrium. The latter is a result first proven by John Nash in 1950, using Brower's fixed point theorem for a continuous function defined using the best reply correspondence.

Proof. We know Γ^n is the set of $S^n \times S^{-n}$ payoff matrices $G^n_{s,t}$, which we shall reparametrise as

$$\left\{ (\tilde{G}^n, g^n) \mid \sum_{t \in \Gamma^{-n}} \tilde{G}^n_{s,t} = 0 \, \forall s \in S^n \right\}.$$

The reparametrisation decomposes the game into a zero-sum part and a residual. Let us define now for each pure strategy $s^n \in S^n$

$$\begin{array}{cccc} z^n_s : & NE & \to & \mathbb{R} \\ & (G, \sigma) & \mapsto & \sigma^n_s + \sum_{t \in \Gamma^{-n}} \left[\tilde{G}^n_{s,t} \prod_{i \neq n} \sigma^i_{t_i} \right] \end{array}$$

Each z^n_s is continuous on E, which implies the continuity of the function defined by

$$\begin{array}{cccc} z^n : & NE & \to & \Gamma^n \\ & (G, \sigma) & \mapsto & (z^n_{s^1}, \ldots, z^n_{s^n}) \end{array}$$

It is left to prove that z is invertible and that its inverse is continuous. To show this let us define $v^n = min\{\alpha \in \mathbb{R} \, : \sum_{s \in S^n} (z^n_s - \alpha)^+ \leq 1\}$

(player 1's equilibrium payoff). Let $\sigma_s^n = (z_s^n - v^n)^+$ and

$$\phi_s{}^n = z_s{}^n - \sigma_s{}^n - \sum_{t\in\Gamma^{-n}} \left[\tilde{G}^n_{s,t} \prod_{m\neq n} \sigma_{t_m}{}^m \right].$$

The resulting vector ϕ is the inverse function of z and fits the given commutative diagram. □

Since the number of strategies is finite, any payoff function is representable by a linear function. The theorem thus covers all possible cases in the setting of finite-dimensional spaces, and indeed the proof relies heavily on the linear structure of payoff functions. An extension theorem that applies to nonlinear payoff functions will be presented in section 2.2.

Γ is homeomorphic to the Euclidean space. The homeomorphism z fails to be differentiable over NE, and so the equilibrium set is not diffeomorphic to Γ. In [Ri], the framework for the discussion of the theory from the differentiable viewpoint is discussed.

2.2 An extension of the structure theorem

The framework that so far has been considered restricts the strategy sets to polyhedra defined by the convex combinations of a finite number of points in Γ. Given $S \subset \mathbb{R}^n$ such that $\sharp S \leq n$, any mapping $S \to \mathbb{R}$ can be represented by a linear function on $\mathbb{R}^n$. Thus in the presence of polyhedral restrictions on strategies, the only relevant payoff functions are the linear ones. As [Pr] points out, the polyhedral structure is an important assumption in the proof of the structure theorem in [KM].

When a wider class of payoff functions is considered, the topology of the space of such functions gains complexity. [Pr] considers strategy sets that are not polyhedral. Instead of $\sharp S^n < \infty$, we consider now a convex and compact set $X^n \subset \mathbb{R}^k$. Let now Γ^n be the set of concave and continuously differentiable functions in player n's own strategy.

Let $\Gamma = \prod_{n\in N} \Gamma^n$ be endowed with the $\mathscr{C}^1$compact-open topology. Its subbase is determined by the sets

$$\{u^n \in \Gamma^n \mid u^n(x) \in E \text{ and } du^n(x^n, n^{-n}) \in E' \ \forall x \in C\}$$

for a compact $C \subseteq \mathbb{R}^k$ and open sets $E \subseteq \mathbb{R}$, $E' \subseteq \mathbb{R}^k$.

The graph of the Nash correspondence is the set NE of points $(g,\sigma) \in \Gamma \times X$ such that σ is a Nash equilibrium of the game g. The following theorem

Theorem 21. *NE is homeomorphic to Γ.*

We now sketch the proof, which is provided in [Pr]. Define the function $\eta\ : NE \to \Gamma$ by $\eta^n(g,\sigma) = g^n + h^n_\sigma g + l^n_\sigma$ for every $n \in N$, and $\eta = (\eta^n)_{n\in N}$. For each n fix $\sigma^n_0 \in X^n$ and define the following functions on X:

$$\begin{array}{rcl} l^n_\sigma(\tau) & = & <\sigma^n, \tau^n> \\ h^n_\sigma g(\tau) & = & <dg_n(\sigma^n,\sigma^{-n}) - dg_n(\sigma^n,\sigma_0^{-n}), \tau^n - \sigma^n> \\ k^n g(\tau) & = & g^n(\tau^n,\sigma_0^{-n}) - \frac{1}{2} <\tau^n,\tau^n> \end{array}$$

Define also

$$\begin{array}{rccc} \phi : & \Gamma & \to & \Gamma \times X \\ & g & \mapsto & (g^n - h^n_\sigma g - l^n_\sigma g)(\sigma) \end{array}$$

where $\sigma^n = argmax\{k^n g(z) :\ \tau \in \Gamma^n\}$ for each $n \in N$. In [Pr] it is proven that both ϕ and η are continuous and mutual inverses.

This new structure theorem might be useful in implementation of economies as games because utility functions are not linear; if an economy is to be encoded as a game and the equivalent of utility is the payoff, it is likely that nonlinear utility functions will be encoded in nonlinear payoffs.

2.3 The differentiable viewpoint

2.3.1 Further concepts of differential topology

Theorem 22 (Poincaré-Hopf)**.** Let $M \subseteq \mathbb{R}^m$ be a compact manifold and w a smooth vector field on M with isolated zeros, such that w points outward at all boundary points. The sum $\sum \iota$ of *indices* at the zeros of w is equal to the Euler number

$$\xi(M) = \sum_{i=0}^{m} (-1)^i rank\, H_i(M),$$

where $H_i(M)$ denotes the ith homology group of M.

The proof for this deep result in differential topology is sketched in [Mi]. Although we have not gone into the concept of index. It is defined in [DG] in the same way as the degree of a mapping, applied now to vector fields. The following lemma (also proven in [Mi]) establishes a link between this and the Brower degree as defined in section 1.2.1.

Lemma 23. The index of w at a regular zero $x \in M$ is either 1 or -1 according to whether the determinant of dw_x is positive or negative.

2.3.2 Regularity of equilibria

Theorems 20 and 21 establish the topological structure of the Nash correspondence. In particular, they imply that for every normal form game of complete information there is a Nash equilibrium, namely a fixed point of the Nash correspondence. The multiplicity of equilibria calls for a refinement of Nash equilibria, and one refinement criterion is that of *regular equilibria*.

We will follow the approach proposed in [Ri], once again embedding individual strategy spaces into the same space $\mathbb{R}^k$. The player's problem is to choose $\sigma^n \in \Sigma^n$ that maximises

$$U^n(\sigma) = \sigma^n \cdot G^n(\sigma).$$

The idea will be to consider the space of strategies with 1 dimension less, namely

$$\Sigma = \prod_{n \in N} \left\{ \sigma^n \geq 0 : \sum_{j=1}^{k-1} \sigma_j^n \leq 1 \right\}$$

of dimension $M = (\sharp N)(k-1)$. us define now $b : \Sigma \to \mathbb{R}^M$ by

$$b_j^n(\sigma) = \sigma_j^n \left[U^n(\sigma^{-n}, s_j^n) - U^n(\sigma) \right]$$

for $j = 1, \ldots, k-1$ and $n \in N$.

The function b is calculated from the basis of player n playing a pure j-strategy. The space has lower dimension than that of the best-reply correspondence because the simplex in $[0,1]^k$ has $k-1$ degrees of freedom. The first lemma in [Ri] is the proof that stable points of the field are precisely the Nash equilibria.

Notice the importance of the reference point in the definition of the best-response vector field. [Ri] uses the equilibrium itself as the reference point. Later on we will see that the reference point in the Walrasian dynamics is given by the endowments themselves.

Definition 24. An equilibrium $\bar{\sigma}$ is *regular* if $|d_\sigma b(\bar{\sigma})| \neq 0$.

In section 1.2.1 we studied regularity in the context of Walrasian equilibria, with respect to the natural projection. In the setting of differentiable games we study regularity properties of equilibria with respect to the Nash field $\vec{b}$.

A mapping $F : \Sigma \times \mathbb{R}_+ \to \mathbb{R}^{\sharp N(k-1)}$ is an *interior approximation* of $\vec{b}$ if

(i) $F(\sigma, 0) = \vec{b}(\sigma)$,

(ii) $f_\lambda : \Sigma \to \mathbb{R}^{\sharp N(k-1)}$ defined by $f_\lambda(\sigma) = F(\sigma, \lambda)$ is a smooth vector field pointing inward at the boundary of Σ, and

(iii) F is continuously differentiable on a neighbourhood of $\Sigma \times \mathbb{R}_+$.

An interior approximation F is *regular* if there exists $\bar{\lambda} > 0$ such that $f_\lambda \equiv F(\sigma, \lambda)$ has only finitely many zeros on Σ, all of them being isolated for any $\lambda \in (0, \bar{\lambda})$. Note that this definition of regularity makes sense in light of the relation between local invertibility and finiteness of the inverse image at a regular point.

Theorem 25. The number of regular equilibria is finite and odd.

Proof. Define the smooth functions $\pi_\delta : int\, \Sigma \to \mathbb{R}$, for $\delta \in (0, k^{-k\sharp N})$, by

$$\pi_\delta(\sigma) = \left(\prod_{n \in N} \left(1 - \sum_{i=1}^{k-1} \sigma_i^n \right) \prod_{i=1}^{k-1} \sigma_i^n \right) - \delta.$$

The gradients of these functions are given by

$$\frac{\partial \pi_\delta(\sigma)}{\partial \sigma_i^n} = \frac{\left(1 - \sum_{j=1}^{k-1} \sigma_j^n - \sigma_i^n\right)(\pi_\delta(\sigma) + \delta)}{\sigma_i^n \left(1 - \sum_{j=1}^{k-1} \sigma_j^n\right)}$$

for all $i = 1, \cdots, k-1$ and $n \in N$. Now, if we had for all $i = 1, \cdots, k-1$ and $n \in N$ that $1 - \sum_{j=1}^{k-1} \sigma_j^n = \sigma_i^n$, this would imply $\sigma_i^n = 1/k$ for each i and

$k.$, such that $\pi_\delta(\sigma) = \prod_{n\in N} k^{-k} - \delta = k^{-k\sharp N} - \delta > 0$ by definition, and so $\sigma \notin \pi_\delta^{-1}(0)$.

From this it follows that 0 is a regular value of π_δ for $\delta \in (0, k^{-kn})$. It follows from [GP] (see section 1.2.1) that $\Pi_\delta = \{\sigma \in int\Sigma \mid \pi_\delta(\sigma) \geq 0\}$ is an $(\sharp N)(k-1)$-differentiable manifold with boundary is $\pi_\delta^{-1}(0)$. Now, as $\delta \downarrow 0$ the compact manifold Π_δ converges to Σ and $\partial\Pi_\delta$ converges to $\partial\Sigma$.

For $\delta \in (0, k^{-k\sharp N})$ define on Π_δ the vector field $\vec{p}: \Pi_\delta \to \mathbb{R}^M$ by $p_i^n(\sigma) = \sigma_i^n k - 1$ for each $i = 1, \cdots, k-1$ and $n \in N$. Its only zero is the uniform distribution $\sigma_i^n = 1/k$. At the boundary $\vec{p}$ points outward (see the proof in [Ri]). According for the Poincaré-Hopf theorem for manifolds with boundary (see [Mi]), the sum of indices at the zeros of $\vec{p}$ equals the Euler characteristic, $\xi(\Pi_\delta)$. By [Mi], the index of $(1/k, \cdots, 1/k)$ is 1, such that $\xi(\Pi_\delta) = 1$ for each $\delta \in (0, k^{-k\sharp N})$.

[Ri] proves that regular equilibria are isolated. From the compactness of Σ it follows there are finitely many, which we denote by $\sigma^1, \cdots, \sigma^Q$. By the implicit function theorem each equilibrium σ^q is continuously approximated by a unique family $\sigma^q(\lambda)_{\lambda>0}$ of zeros of the vector fields f_λ. By continuity of the determinant, there exists $\bar{\lambda} > 0$ such that $\mid d_\sigma f_\lambda(\sigma^q(\lambda)) \mid \neq 0$ for $q = 1, \cdots, Q$ and $\lambda \in [0, \bar{\lambda})$. Let $\lambda_\sigma \in (0, \bar{\lambda})$ such that $\{\sigma^1(\lambda_\sigma), \cdots, \sigma^Q(\lambda_\sigma)\} = f_{\lambda_\sigma}^{-1}(0)$. Choose $\delta > 0$ small enough so that $f_{\lambda_\sigma}^{-1}(0) \subset int\Pi_\delta$ and the vector field f_{λ_σ} points inward at $\partial\Pi_\delta$. This is possible because Π_δ converges to Σ as $\delta \downarrow 0$ and f_{λ_σ} points inward at $\partial\Sigma$.

The set $f_{\lambda_\sigma}^{-1}(0)$ must coincide with the set of zeros of $-f_{\lambda_\sigma}$ which points outward at the boundary of Π_δ. By the Poincaré-Hopf theorem for manifolds with boundary, the indices of zeros of $-f_{\lambda_\sigma}$ must sum to $1 = \xi(\Pi_\delta)$. Since the indices are the sums of signs of

$$|-d_\sigma f_{\lambda_\sigma}(\sigma^q(\lambda_\sigma))|,$$

it follows Q is odd. Making $\lambda \downarrow 0$ transfers the result to $\vec{b}$, and the oddness result follows. □

As [Ri] points out, Σ is not a manifold with simple boundary. The interior approximation technique has to be used to allow the use the Poincaré-Hopf degree theorem for manifolds with simple boundary.

Theorem 26. *NE* can be arbitrarily closely approximated by a differentiable $(\sharp N)K$-manifold, where $K = \sharp S = k^{\sharp N}$.

This result established in [Ri] makes it possible to use the differentiable framework for the geometric study of the best-reply dynamics. However, the approximation manifold in theorem 26 is a differentiable manifold with boundary unlike the Walrasian manifold which has none.

The issue of a reference point leads to a difference in the definition of regularity in the setting of economies and that of games: while initial endowments are the point of reference for the Walrasian dynamics, there is no such initial point in the best-reply dynamics. Different authors have proposed different formulations of the reference point, thus taking approaches to regularity that differ on the reference point of the dynamics.

It makes sense to study the regularity of economies from the viewpoint of the natural projection because of the corollary of Arrow's theorems, mainly that a Pareto optimum (which is the stable point of the Walrasian dynamics) is always associated with a no-trade equilibrium via the natural projection. In the case of games, the main result of [DG] (which we have not listed) equates the index of the best-reply dynamics with the topological degree of the natural projection.

Such result implies different dynamics have essentially same indexes, which are given by the degree of the natural projection. The natural projection thus leads to a topological-analytical constant in both cases, that of games and that of economies. The homeomorphisms in theorems 20 and 21 being in fact homotopic to the natural projection, we can see that the comparative statics in both cases can be examined from the natural projection, even though dealing with different geometric objects.

3 Nash equilibria and Walrasian equilibria

Game theory was developed roughly 100 years later than general equilibrium theory. Until then rational choice in competitive situations had been examined through the lens of general equilibrium theory. However, the need had risen to extend such analysis to frameworks beyond a structure of commodities and prices. With contributions from Cournot, Borel, Von Neumann and Morgenstern, concepts such as that of strategy, strategic independence and expected utility maximisation appeared.

Nash's reformulation of Von Neumann's normal form games and his result of existence defined game theory as is studied nowadays. He showed the breadth of scope of this tool by showing it is possible to analyse a cooperative situation in the framework of non cooperative games. The path to this general theory of rational decision-making had by then distanced from the setting of general equilibrium theory.

However, the two disciplines have been through a parallel epistemic development during the second half of the 20th century. As we have seen so far, a similar geometric approach has led to similar results. In both theories the geometry of the equilibrium set is that of a manifold. Although $WE_{\mathscr{E}}$ is differentiable while NE is not, the latter can be arbitrarily closely approximated by differentiable manifolds.

There is a strong reason for the introduction of implementation. By introducing a central auctioneer, Walras added a fictitious player. The *invisible hand* in the market mechanism remains a black box. Implementation can shed light on it, making institutional aspects of the market visible. Furthermore, it can supply a framework for refining Walrasian equilibria.

3.1 Implementation of Walrasian equilibria as Nash equilibria

A deeper understanding of the relation between strategic optimality and efficient allocation has been reached by relating market equilibria and Nash equilibria. The market mechanism can be seen as a mapping from individual actions to prices and trades, where such mapping is an institution, which becomes the key to the relation. The question is how a market mechanism can be implemented as the equilibrium of a given game such that the equilibria of the game are exactly those of the associated market.

[Hu] states negative results about the attainability of certain allocations through a mechanism of noncooperative games. Later results in [Sv] and [Be] aim at providing limiting conditions on the game mechanism so that the Nash outcome is efficient. Although in general efficient allocations are strategic best responses (*i.e.* Nash equilibria) there are inefficient Nash equilibria. See for instance the generic presence of autarchic Nash equilibria in [DuS].

3.2 Two approaches to strategic market games

3.2.1 No price taking in the trading posts mechanism

The objective is to provide a market mechanism without a central auctioneer. [Gi] proposes a class of mechanisms based on the work by Shapley and Shubik (1977). Given k commodities, the last one will be as usual the numeraire. Each commodity $j = 1, \ldots, k-1$ can be traded against money at the jth trading post. A player's strategy is a pair $(q_j^n, b_j^n) \in \mathbb{R}_+^2$ denoting respectively player n's offer of commodity j and bid (in terms of money) for one unit of the same commodity.

These strategies can be seen as signals. Credibility of these signals imposes the natural restrictions $q^n \leq \omega^n$ on the offer side (where ω^n is the initial endowment for player n) and $\sum_{j=1}^{k-1} b_j^n \leq \omega_k^n$ on the bid side. Player n's strategy set is thus

$$S^n(\omega^n) = \{(q^n, b^n) \in \mathbb{R}_+^{2(k-1)} \mid q_j^n \leq \omega_j^n,\ j = 1, \ldots, k-1, \text{ and } \sum_{j=1}^{k-1} b_j^n \leq \omega_k^n\}.$$

In order for all trade posts to work, the total supply of money available at the jth trading post has to be equal to the price of commodity j times the aggregate supply of commodity at that post. This implies that the price of commodity j is given by

$$p_j = \frac{\sum_n b_j^n}{\sum_n q_j^n}$$

by adding the assumption $x/0 = 0$. The player's final allocation is thus

$$x_j^n = \omega_j^n - q_j^n + \frac{b_j^n}{p_j},\ j = 1, \ldots, k-1$$

while his final balance of money is $x_k^n = \omega_k^n - \sum_{j=1}^{k-1}(b_j^n - q_j^n p_j)$.

The model turns the central auctioneer into k trading post administrators. There is, however, a drawback to this model: as [Gi] points out, in markets where the objects of exchange are symmetric there is no natural numeraire. If a trading post were to be considered for the trading of commodity i against commodity j, the independence of trading posts i and j might lead to an inconsistency of the type $p_{ij}p_{ji} \neq 1$. To solve this inconsistency, another mechanism is proposed: Shapley's "windows model".

This model considers prices that mediate all trades. The trader's signal is now a $k \times k$ matrix whose ij-entry a_{ij}^n indicates the amount of commodity i he is offering in exchange for commodity j. The (central) auctioneer then calculates prices according to the equation of the value of all commodities offered for commodity l with the value of commodity l, *i.e.*

$$\sum_{i=1}^{k}\Big(\sum_{n\in N} a_{ij}^n\Big) = p_l \sum_{j=1}^{k}\Big(\sum_{n\in N} a_{ij}^n\Big),\ j = 1, \ldots, k.$$

Other mechanisms can be considered that may constitute intermediate points between the two introduced here.

3.2.2 Towards efficient equilibria

Price-taking behaviour is a key assumption for efficiency of Nash equilibria in the general equilibrium framework. However, when the agents' strategic influence over prices is taken into account, only an atomless agent space (such as a continuum of uniform agents) will yield efficient outcomes. This is stated in [Gi] as the property that individual price-manipulating power is a decreasing function of the number of agents. Such property corresponds to a result proved in [DS] about the *core* of an economy.

A Pareto optimal allocation is one such that no agent's utility can be improved, except at the cost of another agent's. Although a sense of social justice is sought through this condition, very unfair allocations are also Pareto optimal. That is the case of a dictatorial allocation where everything in the economy goes to one agent. The only way to improve the deprived is to reduce the utility of the dictator. Pareto optimal allocations are not necessarily coalition-safe, except in the case the coalition considered is the whole set of agents.

A core allocation can be seen as a refinement of Pareto optimality in this sense. An allocation is said to be in the core of an economy if no coalition (*i.e.* no subset of agents) can achieve a Pareto-superior allocation (*i.e.* an allocation that strictly improves one agent without harming anyone else) through trade excluding the agents outside of the coalition. In [DS] it is proven that for every finite N the core contains efficient allocations, and as $\sharp N \to \infty$ it "shrinks" to that set.

[DuS] proves a game-theoretic version of this result, namely that whenever the number k of replicas of each player grows to infinity, a subset of the Nash equilibria converges to the Walrasian equilibria of the limit economy. The converging subset is that of equilibria that are also limit points of Nash equilibria of ε-perturbed games as $\varepsilon \to 0$, where the ε-perturbation of the game adds a dummy player that places a bid of $\varepsilon > 0$ in every market.

However, as [Gi] points out, this convergence is generically not true for finite N. In fact, smooth mechanisms yield generically Pareto sub-optimal allocations (including the two mechanisms just presented). The mechanism that is used will play an important role in the relation between $WE_{\mathscr{E}}$ and NE.

3.3 Implementation of Walras equilibria by market games

3.3.1 Encoding economies as games: an example

A finite two player game can be parametrised using matrices A and B of payoffs to players I and II respectively. Given I has n many strategies and II has m many, the matrices have dimension $n \times m$. As [CS] points out, computing Nash equilibria is equivalent to solving *LCP1*, namely finding a non-negative $w \neq 0$ and a non-negative z

such that

$$\begin{aligned} Hw + z &= 1 \\ w^T z &= 0, \end{aligned}$$

where

$$H = \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}$$

Leontief exchange economies are characterised by utility functions describing complementary preference relations, where the marginal utility of one commodity is null once there is more of it than there is of its complement. [CS] the case where $\sharp N = k$ and each agent n has just one unit of commodity n. The economy is described by a square matrix F of size k, where the nth trader has the utility function

$$u^n(x) = min\left\{ \frac{x_i}{f_{ij}} : \ f_{ij} \neq 0 \right\}.$$

An equilibrium for this economy is a non negative price vector $0 \neq \pi \in \mathbb{R}^k$, along with a consumption bundle x yielding a utility level $\beta^n = u^n(x^n)$ for each n, such that for each $j = 1, \ldots, k$

(i) $\beta^j = \frac{\pi_j}{\sum_i f_{ij}\pi_i}$ is well-defined, *i.e.* $\sum_i f_{ij}\pi_i > 0$.

(ii) $\sum_i f_{ij}\beta^j \leq 1$. That is the consumption bundle is feasible.

From the zero entries in H it follows that the set of traders can be split into two groups, where a trader belonging to one group is only interested in commodities offered by a trader from the other group: we call it a two-group Leontief economy. The following two theorems are proven in [CS].

Lemma 27. Let $\beta = (\beta_1, \ldots, \beta_{n+m})$ be the vector of the utility values at equilibrium prices π for the two-group Leontief economy. Then β solves *LCP1* and thus encodes the associated Nash equilibrium.

Lemma 28. Let $w = (w_1, \ldots, w_{n+m}) \neq 0$ be any solution to *LCP1*. Then there exists an equilibrium price vector π such that w is the vector of utility values at these equilibrium prices for the two-group Leontief economy.

[CS] establishes a one-to-one correspondence between Nash and Walras equilibria for a particular case where endowments can actually be encoded. Moreover, the encoding of equilibria relies on the construction of a solution to the dual problem. [CS] points that Cobb-Douglas utility functions can be encoded as zero-sum games via the equivalence between zero-sum games and linear programming (Von Neumann's Minimax Theorem).

However, the one-to-one correspondence is established for a vector β of utilities, which can be obtained from an infinite number of commodity bundles. In fact, from the regular value theorem follows that indifference curves $U^n_c = \{x \in \mathbb{R}^k \mid u^n(x) = c\}$ are $(k-1)$-manifolds. The correspondence is thus established between a vector encoding information about the bundle but also encoding information from a plethora of different bundles. Later on we will have a similar problem when attempting to encode initial endowments as net trades (see section 3.3.3).

[CS] does not interpret the result in terms implementation because its aim is to state a complexity result. However, the two players that interact in the game correspond to the two groups of the Leontief economy. Therefore, even though it goes astray from implementation mechanisms it can bring an understanding of the topological relation between NE and WE. It does so by encoding the utility function as the game matrix, which would suggest a relation between NE and WE instead of $WE_{\mathscr{E}}$.

3.3.2 Mechanisms implementing Walrasian equilibria

While in general equilibrium theory the agents go from one initial state to a stable allocation, in a game such initial state does not exist. The best-response dynamics can be thought of as a transition between strategy profiles over time, but the initial point is not as tangible as it is in the Walrasian dynamics. Because of that, the problem of encoding endowments opens a gap between game theory and general equilibrium theory.

In [BC] a framework for the study of market games is given where a solution to the problem of encoding endowments is proposed. Instead of considering endowments and allocations, the Walrasian equilibrium is formulated in terms of net trades. There is a set N of agents with k commodities where the last one is taken -as usual- as money. Thinking in terms of net trades leads to a new definition of the budget constraint.

Individual preferences are described by a utility function of the form $u^n : \mathbb{R}^k \to \mathbb{R}$ which is increasing in all its arguments and strictly increasing in money. Once a relation f between strategy and demand is established, a payoff function $G^n_u(\sigma) = u^n(f^n(\sigma))$ can be constructed. The Nash equilibrium of the game G^* is thus a strategy profile $(\sigma^n)_{n\in N} \in \Sigma$ such that for every n

$$G^{n*}(\sigma^{n*}, \sigma^{-n*}) \geq G^{n*}(\sigma^n, \sigma^{-n*}).$$

A strategy will be a vector $\sigma^n = (b^n, t^n) \in \Sigma^n$ representing bids (for all commodities except for money) and net trades proposed by player n, where $\Sigma^n \subset \mathbb{R}^{2(k-1)}$ is the strategy set. Let $f : \sigma \mapsto t(\sigma)$ denote the outcome function.

A net trade $t \in (\mathbb{R}^k)^n$ is balanced if $\sum_{n\in N} t^n_j \leq 0$ for every $j = 1, \dots, k$ (meaning commodities have to be disposed of but are not produced). Given $\Sigma = \prod_{n\in N} \Sigma^n$, the outcome function is $f : \Sigma \to F$, where F is the set of balanced trades. A game form (or mechanism) is thus a pair (Σ, f).

Definition 29. A Walrasian equilibrium for an economy $(u^n)_{n\in N}$ is a balanced allocation t^* and a price vector $p^* \in S$ such that $(t^*)^n$ maximises $u^n(t^n)$ subject to $p^* \cdot t^n = 0$ for every n.

There is a difference between this definition and definition 3. In light of the Walrasian definition of efficiency, an equilibrium has to be first a no-waste allocation, namely $\sum_{n\in N} t^n_j = 0$. This is a condition that follows directly when utility functions are strictly increasing. Whether strict monotonicity or lax monotonicity describe economic rationality better is something we shall not discuss. We note, however, that we have gone astray from the ideal no-waste situation analysed in section 1.

Strategies determine allocations and also prices; however, the mechanism transforming strategies into allocations is different from the one transforming strategies into prices. Let $p : \Sigma \to \mathbb{R}^k_+$ be the price function. We observe the following:

1. By construction of S, prices are normalised to make the price of money 1. Because we are assuming that all traders buy at

the same price, unlike for instance Shapley's *W-mechanism* as shown in [Gi], this normalisation makes sense.

2. The old budget constraint

$$\mathscr{B}^n(p,\omega^n) = \{x^n \in \mathbb{R}^k :\ p \cdot x^n \leq p \cdot \omega^n\}$$

   needs a second parameter to be defined, whereas the new budget constraint $\mathscr{B}^n(p) = \{t^n \in \mathbb{R}^k :\ p \cdot t^n \leq 0\}$ only needs one. To allow agents to have different consumption possibilities (as opposed to symmetrical agents), it is enough to fix a different feasible trades for each of them.

3. The vector p is such that $\sum_{n \in N} t^n_j < 0$ implies $p_j = 0$, namely that commodities that are not wanted are worth 0.

Definition 30. A Nash equilibrium of the game form (Σ, f) for an economy $(u^n)_{n \in N}$ is a strategy profile σ^* such that $G^n_u(\sigma^*) \geq G^n_u(\sigma^n, \sigma^{-n*})$ for each $n \in N$ and $\sigma^n \in \Sigma^n$.

[BC] defines another concept, that of strong equilibrium, a type of Nash equilibrium which is robust to coalitions. A relation between the three types of equilibrium is established. For the sake of brevity we will modify the proofs in [BC] in order to examine the equivalence between Nash and Walras equilibria without having recourse to strong equilibria. Two axioms restricting the mechanism will be needed.

Axiom 31 (Unanimity (U)). If $\sigma = (b,t) \in \Sigma$ is such that $b^n = b^m$ for $n, m \in N$, then $p(\sigma) = b^n$. If in addition t is such that $\sum_{n \in N} t^n_j = 0$ for $j = 1, \dots, k-1$ then $t_j(\sigma) = q_j$ for $j = 1, \dots, k-1$.

Axiom 32 (Voluntary trade (VT)). For every $j = 1, \dots, k-1$, $n \in N$ and $\sigma \in \Sigma$, $t^n_j(\sigma) > 0$ implies $p_j(s) \geq b^n_j$.

Lemma 33. Let (Σ, f) be a market form in which **VT** and **U** hold. If (p,t) is a Walrasian equilibrium of the economy $(u^n)_{n \in N}$, there exists σ such that $f(\sigma) = t$.

Proof. Let $b^n = p$ and $q^n = t^n$ for every $n \in N$. Given $f(\sigma) = t$, suppose σ is not an equilibrium of G_u. Let $(\sigma^{n\prime}, \sigma^{-n})$ such that $G^n_u(\sigma^{n\prime}) > G^n_u(\sigma^n)$. Let us call that strategy profile σ'. Let $t' = f(\sigma')$, $p' = p(\sigma')$. We claim $p \cdot t^{n\prime} > p \cdot t^n$. Otherwise t' would be feasible at the price p where no one's utility level decreases and yet that of n strictly increases, contradicting the fact that (p,t) is an equilibrium. $p \cdot t^{n\prime} > 0$ by the preceding claim, and $p' \cdot t^{n\prime} \leq 0$ by the budget constraint inequality. From **U** follows $p = b^n$, and so

$$(p' - b^n) \cdot t^{n\prime} < 0$$

$$\sum_{j=1}^{k-1} (p' - b^n) t_j^{n\prime} < 0$$

$$\sum_{t_j^{n\prime} > 0} (p'_j - b_j^n) t_j^{n\prime} + \sum_{t_j^{n\prime} < 0} (p'_j - b_j^n) t_j^{n\prime} < 0$$

where by **VT** both terms of the sum are positive, leading to a contradiction. □

Axiom 34 (Reactiveness (R)). Let σ be a strategy profile such that $b_j^n > b_j^m$ with $t_j^n(\sigma) > 0$ and $t_j^m(\sigma) < 0$ in each market $j < k$. Then there exists σ' identical to σ except in component r (with r equal to n or m) such that

(i) $t^r(\sigma') = t^r(\sigma)$,

(ii) $(p_j(\sigma') - p_j(\sigma)) \cdot t_j^r(\sigma) < 0$ and

(iii) $p_l(\sigma) = p_l(\sigma')$ for $l < k$ and $l \neq j$.

This axiom means that when the bid of the buyer is greater than that of the seller there is a way for at least one of them to modify the price in a convenient way (depending if he is a buyer or a seller) without affecting either his consumption bundle or the rest of the prices.

Axiom 35 (Strong Bertrand Competition (SBC)). Let σ^{-n} be a strategy profile where all active traders quote the bid p except for trader n. For a given $t^{n\prime}$,

(i) if $t_j^{n\prime} \neq 0$ for some $j < k$ let $I = \{j \mid j = 1, \ldots, k-1 \text{ and } t_j^{n\prime} < 0\}$ and $D = \{\{j \mid j = 1, \ldots, k-1 \text{ and } t_j^{n\prime} > 0\}$. Then for all $b^{n\prime}$ such that $b_j^{n\prime} > p_j$ for $j \in I$ and $b_j^{n\prime} < p_j$ for $j \in D$ there exists q^n such that $t_j^{n\prime} = t_j^n(b^{n\prime}, q^n, \sigma^{-n})$ for $j = 1, \cdots, k-1$ and $p \cdot t^{n\prime} = 0$.

(ii) If $t_j^{n\prime} = 0$ for all $j < k$ there exists $\sigma^{n\prime} = (b^{n\prime}, t^{n\prime})$ with $b^{n\prime} = p$ such that $t^{n\prime}{}_j = t_j^n(\sigma^{n\prime}, \sigma^{-n})$ for $j < k$ and $t_k^n(\sigma^{n\prime}, \sigma^{-n}) = 0$.

This axiom means that by cutting (resp. increasing) the market price a seller (resp. buyer) can transact as much as he wants. Furthermore, by agreeing on the price announcement of all the other agents he always has the possibility of no trade.

Lemma 36. In a market game form in which **VT**, **R** and **SBC** hold any active NE yields a Walrasian allocation.

The lemma is proven in [BC] without using strong equilibria, and so we do not need to rewrite the proof. Let us examine an example: the Shapley-Shubik mechanism shown in section 3.2.1. Recall that an individual strategy is a vector $\sigma^n = (b^n, q^n)$. The mechanism (f, p) is given by

$$p(\sigma) = \left(\left(\frac{\sum_n b_j^n}{\sum_n q_j^n} \right)_{j=1}^{k-1}, 1 \right)$$

$$f(\sigma) = \left(\left(-q_j^n + \frac{b_j^n}{p_j} \right)_{j=1}^{k-1}, -\sum_{j=1}^{k-1} (b_j^n - q_j^n p_j) \right).$$

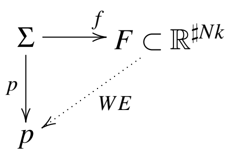
Note that $f(\sigma)$ has to be expressed in terms of net trades t and not in terms of initial endowments ω, such as is formulated in [Gi]; furthermore, trade offers t^n as stated in [BC] are equal to supply offers q^n.

It can be verified that the mechanism does not fulfil the conditions required by [BC], where in return two mechanisms are proposed that fulfil all conditions. In light of the discussion of section 3.2.2, it is worth mentioning that the mechanism is piecewise smooth except over the sets $\{\sigma = (b^n, q^n)_{n \in N} \mid \sum_n q_j^n = 0 \text{ for some } j\}$ and $\{\sigma = (b^n, q^n)_{n \in N} \mid \sum_n b_j^n = 0 \text{ for some } j\}$, where it is discontinuous.

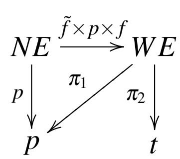
[BC] uses the balanced trade condition with an inequality. If it were to be used with an equality, namely in the case of strictly increasing utility functions, a setting closer to that of Walrasian equilibria could be established. It is worth noting that the Arrow theorems showing the equivalence between Pareto optima and Walrasian equilibria depend on axiom 2 which is abandoned here. Once the axiom is not valid there is a difference between these two concepts.

3.3.3 Nash-Walras equilibria

A mechanism implementing Walrasian equilibria as Nash equilibria is thus determined by functions f and p fulfilling the required conditions. In order to establish the following diagram, we recall that $WE \subseteq \{\mathscr{E}\} \times S \times F$ and $WE_{\mathscr{E}} \subseteq S \times F$. [M] follows the general approach to WE. It proves for instance that WE is closed in $\{\mathscr{E}\} \times S \times F$ where the topology of $\{\mathscr{E}\}$ is the $\mathscr{C}^\infty$compact-open topology.



where F is closed in $\mathbb{R}^{\sharp Nk}$ as the inequality from which it is defined is not strict. WE can be seen as an equilibrium correspondence that associates a set of equilibrium prices to a balanced $t \in F$. The function f induces a function $\tilde{f} : (u^n)_{n\in N} \mapsto (G^n_u)_{n\in N}$ such that, if f satisfies the axioms in [BC] we have the following diagram



We now summarise the results obtained in the following proposition.

Proposition 37. If f satisfies the axioms in [BC] then

$$\begin{array}{cccl} \tilde{f} \times p \times f : & NE & \to & WE \\ & (G, \sigma, \sigma) & \mapsto & ((u^n)_{n\in N}, p, t) \end{array}$$

is a surjective function.

Proof. From lemma 36 it follows $\tilde{f} \times p \times f$ is well-defined and its range is a subset of WE. From lemma 33 it follows the function is surjective. □

As we have seen previously, $WE_{\mathscr{E}}$ is a differentiable manifold for a fixed economy $\mathscr{E} = (u^n)_{n\in N}$ satisfying axioms 1 and 2. The structure of WE, which is the set associated with NE, is related to that of the function space of preferences over the commodity space. Chapter 2 of [M] deals with spaces of preferences, namely the space $\mathscr{C}^r(\mathbb{R}^k)$ for $r \geq 1$. The $\mathscr{C}^r$-compact-open topology (see section 2.2) is complete, separable and metrizable.

We summarise the results in the following diagram

$$\begin{array}{ccccccc}
\Gamma & \xleftarrow{\eta} & NE & \xrightarrow{\tilde{f}\times p\times f} & WE & \xrightarrow{id\times\psi} & \{\mathscr{E}\}\times F \\
\downarrow & & \downarrow & & \downarrow & & \\
G & \xleftarrow{\eta|_G} & NE_G & \xrightarrow{\tilde{f}\times p\times f|_G} & WE_{\mathscr{E}} & \xrightarrow{\psi} & F
\end{array}$$

where both η and ψ are the homeomorphisms respectively in theorems 21 and 9 (ψ is actually a diffeomorphism). We recall that $\Omega = \mathbb{R}^{k(\sharp N)}$ is the space of endowments and Γ is the space of games and can be identified to the space of smooth functions $\{\mathscr{E}$ smooth$\}$ if we restrict NE to smooth games.

The idea would be then to apply once again a theorem similar to 21 to close the cycle with a homeomorphism from $\{\mathscr{E}\}\times F$ to $\{\mathscr{E}\}\cong\Gamma$. We have reached, however, two problems. Having parametrised WE by net trades, the diffeomorphism ψ does not hold anymore. In fact, the setting has changed because of the non-strict concavity of utility functions. Whether a partial structure theorem would still hold for non-strictly concave utilities, or an implementation mechanism for Nash implementation of Walrasian equilibria, would possibly allow to continue in this direction.

Conclusion

As [My] points out, the concept of Nash equilibrium transformed modern economic thought. Not only the results in it are widely used: the concept developed in it has shaped the way interaction is modelled and human behaviour is understood. In particular, its relation to the older economic theory of general equilibrium is of interest in modelling the formation of prices.

Both theories imply the study of dynamic systems in the Euclidean space. In both cases the equilibrium sets, NE and $WE_{\mathscr{E}}$ are manifolds. On one hand, NE is homeomorphic to the underling space of games and can be arbitrarily closely approximated by a differentiable manifold with non-smooth boundary. On the other hand, $WE_{\mathscr{E}}$ is a differentiable manifold diffeomorphic to the underlying space of initial endowments.

General equilibrium theory provides stable points of a dynamic system but does not explain the mechanism through which prices are established. Given the multiplicity of equilibrium prices, game theory can be deemed to provide an answer in explaining how the *invisible hand* works and help refine Walrasian equilibria. Such concern led us to study the implementation of Walrasian equilibria through market games, which amounts to providing restrictions and a relation between the two sets under these restrictions.

In both cases a differentiable approach can be followed. At the regular zeros of a mapping, either the natural projection or the Nash vector field, the topological degree can be defined to study the existence of equilibria. Oddness theorems follow in both cases by proving that the respective function has degree 1. However, the tools applied are different in both cases because the manifolds have different structures: one of them has a boundary (and so the tool that has to be used is not that of degree theory modulo 2).

It is worth noting that the main tool used to obtain this result is a deep result connecting a geometric concept (*i.e.* the Euler characteristic of a manifold) with an analytical concept (i.e. the degree of a function). [DG] relates the index of the Nash field with the topological degree of the natural projection. The fact that the natural projection is the function from which the regularity of Walrasian equilibria is studied could provide a link between the Walrasian dynamics and the best-reply dynamics. Furthermore, it links equilibrium refinements and the geometry of Nash equilibria.

In terms of Nash equilibria, the refinement criterion is *competitiveness* -or efficiency-, which is an economically sound property to impose on equilibria. In terms of Walrasian equilibria, the Nash refinement can shed light on the institutional aspect of the *invisible hand* through making the mechanism explicit and thus explaining the formation of prices in real markets. However, negative results have been established for a wide range of mechanisms. In particular, the fact that smooth mechanisms generically induce Pareto-suboptimal allocations seems to pose a particular problem in the differentiable framework, where a smooth mechanism would have the advantage of linking the best-reply dynamics to the Walrasian dynamics through the chain rule.

The setting of implementation theory seems to challenge the unified vision of efficiency that was established by Arrow in proving the equivalence between Walras equilibria and Pareto optima. In the example we studied, the Nash equilibrium discards disposal-free outcomes. What remains important is the utility-maximisation condition, which is closer to the Pareto optimum condition. Whether a mechanism can go the other way around, or even better preserve the equivalence between both concepts, is something on which we have not come to a positive answer.

Furthermore, the assumptions that lead to implementation are incompatible with the assumptions of the original model. From there two possible paths could be followed: either studying of implementation under conditions allowing to establish (at least) partial versions of the Arrow-Debreu theory or replacing the paradigm of efficiency with that of strategic optimality and trying to bring it as close as possible to efficiency in the new setting. However, we would be looking at the problem in the other way, trying to make Nash equilibria efficient -with respect to a Paretian or Walrasian concept of efficiency- instead of making Walras equilibria Nash implementable.

The outcome function implies an institutional *a priori* aspect of implementation. To a certain extent it could be argued that not much light has been shed on the market mechanism because the problem is only displaced. In particular, in real market situations it may happen that the outcome function is strategy-dependent, which would be a particularly interesting form of endogeneity to consider. However, a result in [Ko] shows that only a dictatorial choice function would be self-consistent in implementing the allocation correspondence. Although it also proposes the use of correspondences instead of functions and the restriction of domains as a way to avoid this negative result, the result in terms of a somewhat democratic market institution is still negative.

Implementation can be seen as a problem of encoding information. In this sense the results from [CS] provide a relation between economies and games. Their aim being to establish a result in hardness of a solution, the fact that the encoding cannot be seen as an implementation mechanism is overlooked. Because the one to one relation between games and economies is provided for a special case of Leontief economies and general two player non zero-sum games, the result would suggest there is not a one to one correspondence between economies and games. In fact, any embedding of one set into the other might even be of Euclidean measure 0. in particular, the one-to-one correspondence proposed is not for commodity bundles but for utiliy levels, which correspond to an infinity of commodity bundles.

Taking that into account, a broader set of economies is considered, namely $WE = \{WE_{\mathscr{E}} \mid \mathscr{E}$ fulfills axioms 1 and 2$\}$. What the suitable conditions on $\mathscr{E}$ would be to establish a relation between WE and NE could be the topic of further studies. Following an approach based on a generalisation of [KM] to the space of strictly concave differentiable functions, our aim was to approach a result linking economies and games. Although the implementation results from [BC] are based on assumptions where the Arrow-Debreu results may not be valid, further results linking these two sets could be found following either weaker versions of the Arrow-Debreu model or weaker implementation results. Even though the Arrow ideal of efficiency might be lost on the way, a ground for the testability of the model can be provided that will make the invisible hand appear.

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